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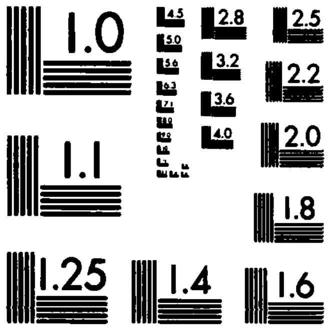

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SOME REMARKS ON THE CREEPING FLOW OF THE SECOND GRADE FLUID

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UNIVERSITY OF WISCONSIN-MADISON
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K. R. Rajagopal[1]

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ABSTRACT

Tanner [1] proved that the Stokes flow solution for the plane flow of the Newtonian fluid is also a solution for the steady plane creeping flow of an incompressible homogeneous fluid of second grade. In this paper we investigate sufficient conditions under which the above theorem can be extended to three dimensional flows. We also investigate certain related questions.

AMS (MOS) Subject Classification: 76A05

Key Words: Stokes flow, creeping flow, Rivlin-Ericksen fluid, normal stress coefficient

Work Unit Number 6 (Miscellaneous Topics)

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SIGNIFICANCE AND EXPLANATION

Tanner's theorem [1] states that the Stokes flow solution for the plane flow of the classical linearly viscous fluid is also a solution for the steady creeping plane flow of an incompressible fluid of second grade. This result greatly simplifies the analysis of several problems in non-Newtonian fluid mechanics. In this paper we discuss conditions under which Tanner's theorem can be generalized to three dimensions.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

SOME REMARKS ON THE CREEPING FLOW OF THE SECOND GRADE FLUID

K. R. Rajagopal[1]

1. INTRODUCTION

Tanner observed [1] that the Stokes flow solution[†] for the velocity field for the plane flow of the Newtonian fluid is a solution for the steady plane creeping flow of an incompressible homogeneous Rivlin-Ericksen fluid of second grade[*]. This simplifies considerably the analysis of several non-Newtonian fluid flow problems. Huilgol [3] proved that under certain conditions the Stokes solution for plane flow is the unique solution to the steady creeping plane flow of a fluid of second grade whose normal stress coefficient $\alpha_1 < 0$. Later Fosdick and Rajagopal [4] proved that under certain conditions in the case of a thermodynamically compatible[**] fluid of second grade, the stokes flow solution is the unique solution to the steady creeping flow of a fluid of second grade in general three dimensional motion.

Tanner's theorem as stated is correct. However, care has to be exercised as the equation governing the creeping flow of a second order fluid is of a higher order than Stokes Equation and in order that one may discuss the

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[†]From henceforth by a solution we shall refer to just a solution for the velocity field. We shall make the tacit understanding that the pressure field in the solution for the second grade fluid flow problem has to be appropriately modified, as is well known.

[*]This result is contained in an earlier work of Giesekus [2].

[**]The fluid is said to be thermodynamically compatible if it meets the Clausius-Duhem inequality in all possible motions and in which the specific Helmholtz free energy is a minimum at equilibrium.

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"solution" to a boundary value problem one needs additional boundary conditions in order that the boundary value problem be well posed.

Tanner's result lead naturally to the following questions:

(1) While Tanner's theorem implies that the Stokes solution is a solution for the plane creeping flow of a fluid of second grade (wherein $\alpha_1 < 0$), is it possible that there exists a steady creeping flow solution to the second grade fluid problem which is not a solution to the Stokes flow problem*? Is it possible that there exists a solution to the creeping flow of a second grade fluid when no solution exists for the corresponding Stokes problem?

(2) Is it possible to generalize Tanner's result to three dimensions? If not what are the sufficient conditions which guarantee the extension of Tanner's results to three dimensions**? Is it possible to explicitly demonstrate a three dimensional Stokes flow solution which does not satisfy the steady creeping flow of a fluid of second grade?

In this paper we shall provide answers to the above questions. We shall explicitly exhibit a steady creeping flow solution to a boundary value problem for an incompressible homogeneous fluid of second grade where no Stokes solution exists for the corresponding boundary value problem.

We shall also exhibit a three dimensional Stokes flow solution which does not satisfy the equations of motion for the steady creeping flow of a fluid of second grade. Of course, the Stokes solution satisfies the equations of

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*While Tanner discusses the possibility of such a solution, he concludes that the Stokes solution "appears to comprise all the useful solutions".

**The analysis of Fosdick and Rajagopal [3] clearly indicates that thermodynamic compatibility is sufficient to guarantee such an extension. However, thermodynamic compatibility is not necessary.

motion for the steady creeping flow of a thermodynamically compatible fluid of second grade. We also investiagate weaker sufficient conditions under which Tanner's result can be extended.

2. ANALYSIS AND REMARKS

The Cauchy stress $\underset{\sim}{T}$ in an incompressible and homogeneous Rivlin-Ericksen fluid of second grade is related to the fluid motion in the following manner (c.f. Truesdell and Noll [5])

$$\underset{\sim}{T} = p\underset{\sim}{1} + \mu\underset{\sim}{A}_1 + \alpha_1\underset{\sim}{A}_2 + \alpha_2\underset{\sim}{A}_1^2 , \tag{1}$$

where μ is the viscosity, α_1 and α_2 material moduli commonly referred to as the normal stress moduli, p is the indeterminate pressure and $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ kinematical tensors defined as (c.f. Rivlin and Ericksen [6])

$$\underset{\sim}{A}_1 = \text{grad } \underset{\sim}{v} + (\text{grad } \underset{\sim}{v})^T , \tag{2$_1$}$$

$$\underset{\sim}{A}_2 = \frac{d\underset{\sim}{A}_1}{dt} + (\text{grad } \underset{\sim}{v})^T\underset{\sim}{A}_1 + \underset{\sim}{A}_1(\text{grad } \underset{\sim}{v}) , \tag{2$_2$}$$

where $\underset{\sim}{v}$ denotes the velocity and $\frac{d}{dt}$ the material time differentiation.

If the fluid is thermodynamically compatible in the sense of the definition in the footnote in the introduction, then the material moduli have to obey the following restriction (c.f. Dunn and Fosdick [7])

$$\mu > 0, \quad \alpha_1 > 0 \quad \text{and} \quad \alpha_1 + \alpha_2 = 0 . \tag{3}$$

It is not our aim here to get into a lengthy discussion about the validity or otherwise of the above restrictions. On the contrary we include the general class of second grade fluids in our analysis and we indicate in addition the consequences of the requirement (3).

On substituting the constitutive expression into the balance of linear momentum

$$\operatorname{div} \underset{\sim}{T} + \rho \underset{\sim}{b} = \rho \frac{d\underset{\sim}{v}}{dt} ,$$

one obtains that

$$\mu \Delta \underset{\sim}{v} + \alpha_1 (\Delta \underset{\sim}{w} \times \underset{\sim}{v}) + \alpha_1 \Delta \underset{\sim}{v}_t + (\alpha_1 + \alpha_2)\{\underset{\sim}{A}_1 \Delta \underset{\sim}{v} + 2\operatorname{div}[(\operatorname{grad} \underset{\sim}{v})(\operatorname{grad} \underset{\sim}{v})^T]\}$$
$$- \rho(\underset{\sim}{w} \times \underset{\sim}{v}) - \rho \underset{\sim}{v}_t = \operatorname{grad} P , \qquad (4)_1)$$

where

$$P = p - \alpha_1 \underset{\sim}{v} \cdot \Delta \underset{\sim}{v} - \frac{1}{4}(2\alpha_1 + \alpha_2)|\underset{\sim}{A}|^2 + \frac{1}{2}\rho|\underset{\sim}{v}|^2 + \rho\phi , \qquad (4)_2$$

and

$$\underset{\sim}{w} = \operatorname{curl} \underset{\sim}{v} .$$

In the above equations Δ denotes the Laplacian, $|\underset{\sim}{v}|$ denotes the usual innerproduct norm of $\underset{\sim}{v}$ and $|\underset{\sim}{A}|$ the trace norm of $\underset{\sim}{A}$. Also in deriving the above equations we have made use of the fact that

$$\operatorname{div} \underset{\sim}{v} = 0 , \qquad (5)$$

since the fluid is incompressible and we have assumed that the body force field is conservative and hence derivable from a potential, i.e., $\underset{\sim}{b} = -\operatorname{grad} \phi$.

In the case of steady motion, if one ignores the inertial terms, the above equation reduces to

$$\mu \Delta \underset{\sim}{v} + \alpha_1 (\Delta \underset{\sim}{w} \times \underset{\sim}{v}) + (\alpha_1 + \alpha_2)\{\underset{\sim}{A}_1 \Delta \underset{\sim}{v} + 2\operatorname{div}[(\operatorname{grad} \underset{\sim}{v})(\operatorname{grad} \underset{\sim}{v})^T]\} = \operatorname{grad} \hat{P} \qquad (6)_1$$

where

$$\hat{P} = p - \alpha_1 \underset{\sim}{v} \cdot \Delta \underset{\sim}{v} - \frac{1}{4}(2\alpha_1 + \alpha_2)|\underset{\sim}{A}_1|^2 . \qquad (6)_2$$

We eliminate $\hat{P}$ by taking the curl of equation $(6)_1$ to obtain

$$\mu \Delta \underset{\sim}{w} + \alpha_1 \operatorname{curl}(\Delta \underset{\sim}{w} \times \underset{\sim}{v}) + (\alpha_1 + \alpha_2)\operatorname{curl}\{\underset{\sim}{A}_1 \Delta v$$
$$+ 2\operatorname{div}[(\operatorname{grad} \underset{\sim}{v})(\operatorname{grad} \underset{\sim}{v})^T]\} = 0 . \qquad (7)^*$$

---

*It is obvious from equation (7) that in the case of a thermodynamically compatible fluid of second grade the Stokes solution $\Delta \underset{\sim}{w} = 0$ is also a solution to (7). In [4], it is shown that it is the unique solution to (7) provided certain conditions are met.

It is important to note that equation (4), if of higher order than the equation governing Stokes flow. Thus, in order that the problem be well posed, one needs boundary conditions in addition to the boundary conditions for the Stokes problem. Tanner's theorem does not address this question. Tanner's theorem states "Any plane creeping Newtonian velocity field with given velocity boundary conditions is also a solution for the second order incompressible fluid with the same boundary conditions". He does not make any restriction on the additional boundary condition that is required for the creeping second order fluid flow problem.

We shall now consider the first question posed in the introduction. We exhibit a solution to a creeping flow problem for second grade fluids which is exact when a corresponding solution does not exist for the Stokes flow problem.

Let us consider the flow of a second grade fluid past a porous plate which is subject to suction (Figure 1). We then seek a solution of the form

$$u = u(y) \; , \tag{9$_1$}$$

$$v = -v_0 \; , \quad v_0 > 0 \; , \tag{9$_2$}$$

$$w = 0 \; , \tag{9$_3$}$$

which automatically satisfies (5). It then follows from (9) that

$$\frac{d^3u}{dy^3} - \frac{\alpha_1 v_0}{\mu} \frac{d^4u}{dy^4} = 0 \; . \tag{10}$$

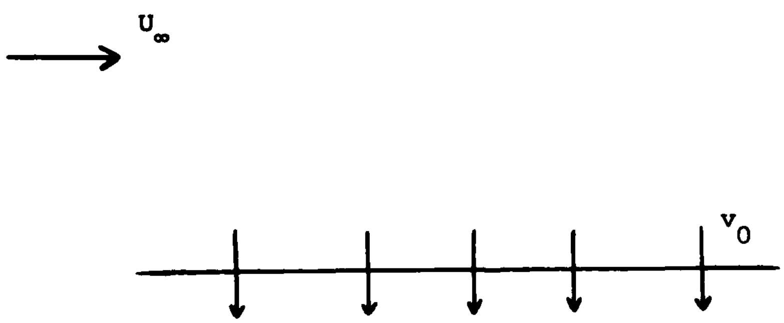

Figure 1

The appropriate boundary conditions are

$$\underset{\sim}{u}(0) = \underset{\sim}{0} \quad \text{and} \quad \underset{\sim}{u} \to U_\infty \underset{\sim}{i} \quad \text{as} \quad y \to \infty, \quad \frac{d\underset{\sim}{u}}{dy} \to 0 \quad \text{as} \quad y \to \infty \,. \tag{11}$$

Let us suppose we are concerned with an incompressible fluid of second grade whose normal stress moduli α_1 satisfies $\alpha_1 < 0$. Further, let us suppose that $\mu > 0$. Then, it follows that

$$\frac{\alpha_1 v_0}{\mu} < 0 \,.$$

It is trivial to verify that

$$u(y) = U_\infty(1 - e^{my}) \,,$$

with

$$m = \frac{\mu}{\alpha_1 v_0} < 0$$

is a solution to (10). A similar result can be established by considering the blowing problem for an incompressible fluid of second grade wherein $\alpha_1 > 0$.

The above suction boundary value problem can be solved exactly (c.f. Schlichting [8]) in the case of the Newtonian fluid even when inertial effects are included. The appropriate equation in this case is

$$\frac{d^3u}{dy^3} + \frac{\rho v_0}{\mu}\frac{d^2u}{dy^2} = 0 \,,$$

subject to the boundary conditions (11). It follows that the solution in this case is

$$u(y) = U_\infty\left(1 - e^{-\frac{\mu}{\rho v_o} y}\right) .$$

However, if one ignores the inertial effect completely the problem has no solution which would satisfy the boundary conditions. However, we have shown

that the corresponding problem in the case of a fluid of second grade does have a solution.

We now proceed to answer the second question, namely we exhibit a three dimensional Stokes flow solution which does not satisfy the steady creeping flow of an incompressible homogeneous fluid of second grade. Thus Tanner's theorem cannot be generalized as it stands into three dimensions.

Consider the Stokes flow of a Newtonian fluid between two parallel plates rotating about different axes with constant but differing angular velocities (see Figure 2). The appropriate boundary conditions for the velocity field are

$$u = \frac{\Omega_2 a}{2} - \Omega_2 y, \quad v = \Omega_2 x, \quad w = 0 \quad \text{at} \quad z = \frac{h}{2},$$

$$u = -\frac{\Omega_1 a}{2} - \Omega_1 y, \quad v = \Omega_1 x, w = 0 \quad \text{at} \quad z = -\frac{h}{2},$$

and

$$u \to \mp\infty, \quad v \to \pm\infty \quad \text{as} \quad x, y \to \pm\infty .$$

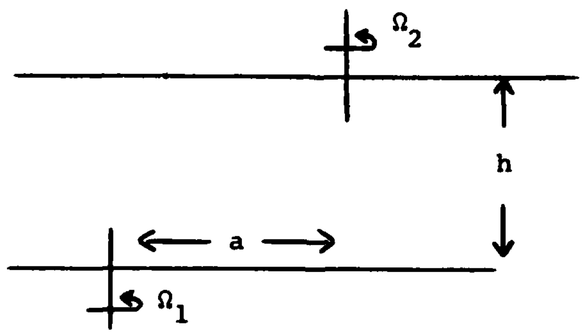

Figure 2

It is pretty straightforward to verify that the unique solution to the Stokes problem is given by

$$u = -\left[\frac{\Omega_2 - \Omega_1}{h} z + \frac{\Omega_2 + \Omega_1}{2}\right](y - \frac{a}{h} z) , \qquad (8)_1$$

$$v = \left[\frac{\Omega_2 - \Omega_1}{h} z + \frac{\Omega_2 + \Omega_1}{2}\right] x , \qquad (8)_2$$

$$w = 0 , \qquad (8)_3$$

where u,v and w denote the x,y and z components of the velocity $\underset{\sim}{v}$ respectively.

It is now easy to verify that $(8)_{1,2,3}$ do not satisfy the equation which governs the creeping flow of a second grade fluid. However, if the fluid is thermodynamically compatible, then since $\alpha_1 + \alpha_2 = 0$, it follows that $(8)_{1,2,3}$ in fact satisfies (7)*. The above result clearly shows that Tanner's result cannot be generalized without some restrictions. The following two results follow trivially from (7).

<u>Theorem 1</u>: A sufficient condition that the Stokes solution satisfy the equations for the steady creeping flow of an incompressible homogeneous fluid of second grade is that $\alpha_1 + \alpha_2 = 0$†.

<u>Theorem 2</u>: A sufficient condition that the Stokes solution satisfy the equations for the steady creeping flow of an incompressible homogeneous fluid of second grade is that

$$\underset{\sim}{A}_1 \Delta \underset{\sim}{v} + 2\mathrm{div}[(\mathrm{grad}\ \underset{\sim}{v})(\mathrm{grad}\ \underset{\sim}{v})^T] = \mathrm{grad}\ \psi ,$$

for some scaler field ψ.

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*This result is in fact a consequence of the analysis of Fosdick and Rajagopal [3].

†It has been brought to may attention by Prof. Millard Johnson that the condition that $\alpha_1 + \alpha_2 = 0$ implies that the ratio of the normal stress differences be -1/2 which is the case in the co-rotational model of Goddard and Miller [10]. The condition that the ratio of the normal stress differences be -1/2 is also qualitatively in keeping with the model of Johnson and Segalman [11].

Theorem 1 does not require that the fluid be thermodynamically compatible since α_1 is not required to be non-negative. It is easy to show by extending Huilgol's analysis that the above solution is unique when $\alpha_1 < 0$ and $\alpha_1 + \alpha_2 = 0$. When $\alpha_1 > 0$ and $\alpha_1 + \alpha_2 = 0$, the uniqueness results of Fosdick and Rajagopal apply.

The class of 3 dimensional flows wherein the hypothesis of Theorem 2 apply are non-trivial. Consider for instance the steady flow between two infiite parallel plates rotating with the same constant angular velocity Ω about non-coincident axes (c.f. Rajagopal [8]). In this case the velocity field is of the form

$$u = -\Omega[y - g(z)] ,$$

$$v = \Omega[x - f(z)] ,$$

$$w = 0 .$$

It is pretty straightforward to show that

$$\underset{\sim}{A}_1 \Delta \underset{\sim}{v} + 2\mathrm{div}[(\mathrm{grad}\ \underset{\sim}{v})(\mathrm{grad}\ \underset{\sim}{v})^T] = \mathrm{grad}\ \psi ,$$

where

$$\psi = \frac{1}{2} [f'^2 + g'^2] .$$

<u>Theorem 3</u>: A necessary and sufficient condition that Stokes solution satisfy the equations for the steady creeping flow of an incompressible fluid of second grade is that

$$(\alpha_1 + \alpha_2)\{\underset{\sim}{A}_1 \Delta \underset{\sim}{v} + 2\mathrm{div}[(\mathrm{grad}\ \underset{\sim}{v})(\mathrm{grad}\ \underset{\sim}{v})^T]\} = \mathrm{grad}\ \psi ,$$

for some scaler field ψ.

<u>Acknowledgement</u>. I would like to thank Professors Millard Johnson, Robert Bird and Arthur Lodge for useful discussions regarding the above work.

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